Topology HW 7 **Fall 2025**

1. Let X be locally compact and Hausdorff. Prove that $(X\times\mathbb{R})^*$, the one-point compactification of $X\times\mathbb{R}$ is homeomorphic to ΣX^*, the suspension of the one point compactification of X. What does this say for $X=\mathbb{R}^n$?

2. Consider the suspension-loop adjunction $\Sigma : \mathbf{Top} \leftrightarrows \mathbf{Top} : \Omega$. The unit and counit of this adjunction are maps

$$\eta_X : X \to \Omega\Sigma X \text{ and } \epsilon_Y : \Sigma\Omega Y \to Y$$

Describe these maps and verify the two "triangle identities"

$$\epsilon_{\Sigma X}\circ\Sigma(\eta_X)=\mathrm{id}_{\Sigma X} \text{ and } \Omega(\epsilon_Y)\circ\eta_{\Omega Y}=\mathrm{id}_{\Omega Y}\,.$$

3. Consider the points $e=1$, $n=i$, $w=-1$, and $s=-i$ on S^1 and let $U=S^1\setminus\{e,w\}$ and $V=S^1\setminus\{n,s\}$. Then S^1 is the pushout

$$\begin{array}{ccc} U\cap V & \xrightarrow{j_U} & U \\ {\scriptstyle j_V}\downarrow & & \downarrow \\ V & \longrightarrow & S^1 \end{array}$$

Use the groupoid version of SVK to compute $\Pi_1(S^1)$ and $\pi_1(S^1,1)$.

4. The map $\exp:\mathbb{C}\to\mathbb{C}^*$ is a covering map. Prove that there cannot exist a function $g:\mathbb{C}^*\to\mathbb{C}$ with $\exp\circ g=\mathrm{id}_{\mathbb{C}^*}$.

5. Consider the diagram $S^1\leftarrow S^1\leftarrow S^1\leftarrow\cdots$ where every arrow is the map $z\mapsto z^p$ for a prime p. The limit of this diagram, call it X_p, comes with maps to each S^1 in the picture. Decide whether these maps $X_p\to S_1$ are covering maps and (if so) analyze the relevant groups.

6. Let $X=S^1\vee S^1$.

(a) Classify all two-sheeted covers of X up to isomorphism.

(b) Now take two of these two-sheeted covers $E\to X$ and $E'\to X$ and draw their pullback, which is a new cover of X.

$$\begin{array}{ccc} E'' & \longrightarrow & E' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ E & \xrightarrow[p]{} & X \end{array}$$

(c) The covers in your pullback correspond to subgroups H and K of G, the free group on two generators. What is the index of $H\cap K$ in G?

7. The fundamental group of the Klein bottle K is given by $\langle r,s : rsr=s\rangle$. If we add the relations r^n and s^2 we get the Dihedral group D_n. So, K admits a cover with Galois group D_n. Describe it and describe the cover when we only add the relation s^2.