1. Carefully prove:

(a) The quotient of $[0,1]$ obtained by identifying 0 and 1 is homeomorphic to S^1.

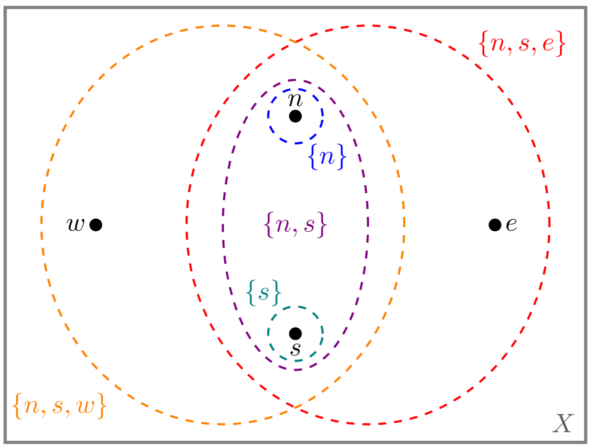
(b) That the standard model of the torus obtained by identifying opposite sides of a unit square is homeomorphic to $S^1 \times S^1$.

2. Let $X = [0,1]$. Let $S \subseteq X$ be any set that is not open and let τ be any topology on X that contains all the sets that are open in the usual topology together with (at least) the set S. Prove that X with the topology τ cannot be compact.

3. Let $X = \{n, s, e, w\}$ with the topology

$$\{\emptyset, \{n\}, \{s\}, \{n,s\}, \{n,s,e\}, \{n,s,w\}, \{n,s,e,w\}\}$$

Here is a picture:



Even though X only has four points, there are interesting paths in X. For example, $\alpha : [0,1] \to X$ defined by

$$\alpha(t) = \begin{cases} w & \text{if } 0 \le t \le \frac{1}{3} \\ n & \text{if } \frac{1}{3} < t < \frac{2}{3} \\ e & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

is a path from w to e.

(a) Is X path connected?

(b) Is the map $X \to S^1$ defined by

$$w \mapsto (-1,0) \quad n \mapsto (0,1) \quad e \mapsto (1,0) \quad s \mapsto (0,-1)$$

continuous?

(c) Can you define a map $S^1 \to X$ that is not constant?

4. According to The Wikipedia article on Expansive Homeomorphism:

Definition. If (X, d) is a metric space, a homeomorphism $f : X \to X$ is said to be expansive if there is a constant $\epsilon > 0$, called the *expansivity constant*, such that for every pair of points $x \neq y$ in X there is an integer $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) \geq \epsilon.$$

Note that in this definition, n can be positive or negative. The Wikipedia article goes on to say *"The space X is often assumed to be compact, since under that assumption expansivity is a topological property."* Let's make this remark perfectly clear.

(a) Show that the two metrics on $\mathbb{R}$ defined by $d_1(x, y) = |x - y|$ and $d_2(x, y) = |e^x - e^y|$ are both compatible with the ordinary topology on $\mathbb{R}$, but the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ is expansive for d_2 and not expansive for d_1.

(b) Suppose (X, d) is a compact. Prove that if a homeomorphism $f : X \to X$ is expansive for one metric compatible with the topology on X then it is expansive for every metric compatible with the topology on X.

5. Prove or disprove:

(a) Hausdorff is a homotopy invariant.

(b) Compact is a homotopy invariant.

(c) Path connected is a homotpy invariant.

6. Suppose X is Hausdorff and $f : X \to X$. Prove that the set $\text{Fix}(f) = \{x \in X : f(x) = x\}$ is closed.