Instructions. These are some fun problems mixing product topology and dynamics, with a few optional challenge items. Work on them over the next week or so. Aim to have this **checked off by September 25**.

- I will not collect full written solutions to everything. We'll discuss a few in class; others may reappear on future sets or exams.
- Discuss freely—working in a small group is encouraged. Please avoid AI tools; they tend to hand you the answers, which spoils the fun and the thrill of discovery.
- By Sept 25, choose *either*: (a) a quick oral check during office hours; (b) email me a one-page summary of which problems you solved and any questions you still have; or (c) if you want written feedback, submit solutions to problems 2b, 3, 7b, 8b, and 12a.

The Cantor Space

Consider $\{0,1\}$ with the discrete topology. The set $C = \{0,1\}^{\mathbb{N}}$ of binary sequences with the product topology is called the Cantor space.

1. Prove the following facts about the Cantor space.

(a) $C \cong C \times C$

(b) $C \cong C \sqcup C$

(c) Prove that C has no isolated points.

(d) C is *totally disconnected* meaning every connected component of C is a singleton.

(e) C is metrizable. (Hint: use $d(x,y) = 2^{-\min\{n:\, x_n \neq y_n\}}$.)

2. Let $s : C \to C$ defined by $s(x_1, x_2, \ldots) = x_2, x_3, \ldots$ be the backwards shift map. Let s^n denote the n-th iterate of s. A point x is *periodic with period n* iff $s^n(x) = x$. The *least period* of a periodic point x is the smallest positive integer n for which $s^n(x) = x$.

(a) Prove that s is continuous.

(b) Prove that the periodic points are dense in C.

(c) Prove that the number of points with least period n are equal to

$$\sum_{d|n} \mu(d) 2^{\frac{n}{d}}$$

where μ is the Möbius function from number theory.

(d) Can you find a point $x \in C$ that has a dense orbit?

3. A homeomorphism $f : X \to X$ is *topologically mixing* if and only if for all nonempty open sets U, V there exists an integer $N \in \mathbb{Z}$ so that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

If $f : X \to X$ is any continuous function (not necessarily a homeomorphism), you can say f is *topologically forward mixing* if and only if for all nonempty open sets U, V there exists an integer $N \in \mathbb{N}$ so that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Prove that the shift map s is topologically forward mixing.

4. A homeomorphism $f : X \to X$ on a metric space is *expansive* if and only if there exists a $\delta > 0$ so that for all $x, y \in X$ there exists an $N \in \mathbb{N}$ so that $d(f^n(x), f^n(y)) > \delta$. Prove that the shift map s is expansive.

5. (Optional challenge) Look up the definition of topological entropy for a dynamical system and compute it for the shift map.

The 2-adic integers

Consider the following diagram of discrete topological spaces

$$\mathbb{Z}/2\mathbb{Z} \xleftarrow{\pi_1} \mathbb{Z}/4\mathbb{Z} \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_{n-1}} \mathbb{Z}/2^n\mathbb{Z} \xleftarrow{\pi_n} \mathbb{Z}/2^{n+1}\mathbb{Z} \xleftarrow{\pi_{n+1}} \cdots$$

where π_n is reduction mod 2^n. The 2-adic integers are defined to be the *limit* of this diagram. That is the subspace of the product defined as follows:

$$\mathbb{Z}_2 \ := \ \lim\{\mathbb{Z}/2^{n+1} \xrightarrow{\pi_n} \mathbb{Z}/2^n\mathbb{Z}\} \ = \ \Big\{(x_n)_{n\geq 1} \in \prod_{n\geq 1} \mathbb{Z}/2^n\mathbb{Z} : \ x_n \equiv x_{n+1} \pmod{2^n}\Big\}.$$

So $(1, 3, 3, 11, 11, 43, 107, \ldots)$, for example, could be the beginning of a typical sequence in $\mathbb{Z}_2$.

6. (a) Show $\mathbb{Z}_2$ is closed in $\prod \mathbb{Z}/2^n\mathbb{Z}$.

(b) Show $\mathbb{Z}_2$ is totally disconnected.

(c) Prove that $\mathbb{Z}_2$ has no isolated points.

(d) Prove that $\mathbb{Z}_2$ is metrizable.

7. Consider the map $T : \mathbb{Z}_2 \to \mathbb{Z}_2$, $T(x) = x + 1$.

(a) Define a metric on $\mathbb{Z}_2$ by $d(x, y) = 2^{-\min\{n:\ x_n \neq y_n\}}$. Prove that T is a homeomorphism and an isometry.

(b) Prove that the orbit of every point is dense in $\mathbb{Z}_2$ and that T has no periodic points.

8. Define a map $\Phi : C \to \mathbb{Z}_2$ from the Cantor Set to the 2-adic integers as follows:

$$\Phi_{\text{seq}}\big((x_0, x_1, \ldots)\big) \;=\; (r_k)_{k\geq 1},$$

where

$$r_k \;=\; \left(\sum_{n=0}^{k-1} x_n\, 2^n\right) \mod 2^k \;\in\; \mathbb{Z}/2^k\mathbb{Z}.$$

So, for example $\Phi(1,1,0,1,0,0,\ldots) = (1,3,3,11,\ldots)$. To see this, look at

$$\begin{aligned} 1 \times 2^0 &\mod 2 = 1 \\ 1 \times 2^0 + 1 \times 2^1 &\mod 4 = 3 \\ 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 &\mod 8 = 3 \\ 1 \times 2^0 + 1 \times 2^1 + 0 \times 2^2 + 1 \times 2^3 &\mod 16 = 11 \end{aligned}$$

(a) Prove Φ is a homeomorphism.

(b) Define the *odometer* $\tau : C \to C$ to be $\tau = \Phi^{-1} T \circ \Phi$, i.e. the transport of the map T via the isomorphism Φ:

$$\begin{array}{ccc} C & \xrightarrow{\tau} & C \\ \downarrow \Phi & & \downarrow \Phi \\ \mathbb{Z}_2 & \xrightarrow{T} & \mathbb{Z}_2 \end{array}$$

Explain how τ works explicitly as a map from $C \to C$.

9. Prove that the odometer τ is not topologically mixing, and is not expansive.

10. (Optional challenge) Look up the definition of topological entropy for a dynamical system and compute it for the odometer.

The torus

11. The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n,m)\cdot(x,y) \mapsto (x+n, y+m)$ defining an equivalence relation $(x,y) \sim (x+n, y+m)$ for $(n,m) \in \mathbb{Z}^2$. Define the torus T^2 to be the quotient

$$T^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

Let $p : \mathbb{R}^2 \to T^2$ be the quotient map $(x,y) \mapsto [(x,y)]$.

(a) Show that the map $\psi : \mathbb{R}^2 \to S^1 \times S^1$ defined by $\psi : (x,y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ induces a homeomorphism $\mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{\cong} S^1 \times S^1$.

(b) For any 2 by 2 matrix with integer entries $A \in M_2(\mathbb{Z})$, define $f_A : T^2 \to T^2$ by $f_A([v]) = [Av]$.

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \downarrow p & & \downarrow p \\ T^2 & \xrightarrow{f_A} & T^2 \end{array}$$

Check the details to understand why f_A is well defined and continuous.

12. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $f = f_A$.

(a) Show that $f : T^2 \to T^2$ is a homeomorphism.

(b) Find the periods of $[(0,0)], \left[\left(0, \frac{1}{4}\right)\right]$, and $\left[\left(\frac{1}{5}, \frac{2}{5}\right)\right]$.

(c) Compute the eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$ and corresponding eigenvectors $v^u, v^s \subset \mathbb{R}^2$. Define lines $E^u = \mathbb{R}v^u$ and $E^s = \mathbb{R}v^s$.

(d) For any point $[x] \in T^2$, choose a lift $\tilde{x} \in \mathbb{R}^2$ and and define the *unstable/stable lines through* $[x]$ by

$$W^u([x]) = p(\tilde{x} + E^u), \qquad W^s([x]) = p(\tilde{x} + E^s).$$

Show these are independent of the choice of lift and f-invariant: $f(W^u([x])) = W^u(f([x]))$ and $f^{-1}(W^s([x])) = W^s(f^{-1}([x]))$.

(e) Prove that f_A is expansive.

13. (Optional challenge) Again, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $f = f_A$.

(a) Show that periodic points of f are dense in T^2.

(b) Show that f is topologically mixing.