A little algebraic topology

1. A retraction (of X onto A) is a continuous map $r: X \to A$ with $ri = \mathrm{id}_A$ where $i: A \hookrightarrow X$ is the inclusion:

$$A \xrightarrow{i} X \xrightarrow{r} A$$

$$id_A$$

Let $D^{n+1}=\{x\in\mathbb{R}^{n+1}:|x|\leq 1\}$ and $S^n=\{x\in\mathbb{R}^{n+1}:|x|=1\}$. Prove that there is no retraction of D^{n+1} onto S^n .

A little set theory

2. Let X and Y be sets and $f: X \to Y$. Some people define f to be surjective iff

for all
$$y \in Y$$
 there exists $x \in X$ with $fx = y$

and define f to be injective iff

for all
$$x, x' \in X$$
 if $x \neq x'$ then $fx \neq fx'$.

Use these definitions of surjective and injective to prove:

- (a) f is surjective iff f is right-cancellable.
- (b) f is injective iff f is left-cancellable.
- (c) f is surjective iff $Set(Y,Z) \xrightarrow{f^*} Set(X,Z)$ is injective for all Z.
- (d) f is injective iff $Set(Z, X) \xrightarrow{f_*} Set(Z, Y)$ is injective for all Z.

(danger below this line)

- (e) f is surjective iff f is right-invertible.
- (f) f is injective iff f is left-invertible.
- (g) f is injective iff $\mathsf{Set}(Y,Z) \overset{f^*}{\to} \mathsf{Set}(X,Z)$ is surjective for all Z.
- (h) f is surjective iff $Set(Z, X) \xrightarrow{f_*} Set(Z, Y)$ is surjective for all Z.
- (i) If f has a unique left inverse then it is (two-sided) invertible.
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A little point-set topology

- **3.** Let (X,d) be a metric space. Define a set $U \subset X$ to be *open* iff for all $x \in U$ there exists $\epsilon > 0$ so that for all $x' \in X$ if $d(x,x') < \epsilon$ then $x' \in U$.
 - (a) Prove that the arbitrary union of open sets is open.
- (b) Prove that the intersection of finitely many open sets is open.
- (c) Prove \emptyset and X are both open.
- **4.** A function $f: X \to Y$ between two metric space is continuous iff for all $x \in X$ and for all $\epsilon > 0$ there exists a $\delta > 0$ so that if $d_X(x, x') < \delta$ then $d_Y(fx, fx') < \epsilon$.. Prove that f is continuous iff $f^{-1}(U)$ is open in X whenever U is open in Y.