Definition 1 (Basis)**.** *Let X be a set. A collection $\mathcal{B}$ of subsets of X is a* basis *for a topology on X iff:*

(i) For each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

(ii) If $x \in A \cap B$ with $A, B \in \mathcal{B}$, then there is $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

The sets $B \in \mathcal{B}$ are called basic open sets*. For $x \in X$, the sets $B \in \mathcal{B}$ with $x \in B$ are the* basic open neighborhoods *of x.*

Assume $\mathcal{B}$ satisfies (i) and (ii). A set $U \subseteq X$ is "open in the topology generated by $\mathcal{B}$" iff for every $x \in U$ there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$. The sets that are open in the topology generated by $\mathcal{B}$ do in fact comprise a topology which I will denote by $\tau_{\mathcal{B}}$ and which may be called "the topology generated by the basis $\mathcal{B}$."

Proposition 1. *The collection $\mathcal{T}_{\mathcal{B}}$ is a topology on X, it contains $\mathcal{B}$, and it is the coarsest topology on X that contains $\mathcal{B}$.*

Proof. First, the topology axioms: $\emptyset \in \mathcal{T}_{\mathcal{B}}$ since it vacuously satisfies the condition to be open. For X, condition (i) gives for each $x \in X$ some $B \in \mathcal{B}$ with $x \in B \subseteq X$, hence $X \in \mathcal{T}_{\mathcal{B}}$.

Let $\{U_i\}_{i \in I} \subseteq \mathcal{T}_{\mathcal{B}}$ and put $U = \bigcup_i U_i$. If $x \in U$, then $x \in U_j$ for some j, so there exists $B \in \mathcal{B}$ with $x \in B \subseteq U_j \subseteq U$. Hence $U \in \mathcal{T}_{\mathcal{B}}$.

For finite intersections, let $U, V \in \mathcal{T}_{\mathcal{B}}$ and $x \in U \cap V$. Choose $A, B \in \mathcal{B}$ with $x \in A \subseteq U$ and $x \in B \subseteq V$. By (ii) there exists $C \in \mathcal{B}$ with $x \in C \subseteq A \cap B \subseteq U \cap V$, so $U \cap V \in \mathcal{T}_{\mathcal{B}}$.

To see that the sets in $\mathcal{B}$ are open in the topology generated by $\mathcal{B}$, observe that if $x \in B \in \mathcal{B}$, then $x \in B \subseteq B$, so $B \in \mathcal{T}_{\mathcal{B}}$.

To see that $\tau_{\mathcal{B}}$ is the coursest topology containing $\mathcal{B}$, let $\mathcal{T}$ be any topology on X with $\mathcal{B} \subseteq \mathcal{T}$. For $U \in \mathcal{T}_{\mathcal{B}}$ and each $x \in U$ choose $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Then

$$U = \bigcup_{x \in U} B_x,$$

a union of sets in $\mathcal{B} \subseteq \mathcal{T}$, hence $U \in \mathcal{T}$. ☐

Remark 1. *Also, $\mathcal{T}_{\mathcal{B}} = \bigcap \{\mathcal{T} \mid \mathcal{T}$ is a topology on X with $\mathcal{B} \subseteq \mathcal{T}\}$.*

Proof. From above, $\mathcal{T}_{\mathcal{B}}$ is contained in every topology that contains $\mathcal{B}$ so it's contained in the intersection. Also, $\mathcal{T}_{\mathcal{B}}$ is a topology that contains $\mathcal{B}$ so the intersection is contained in $\mathcal{T}_{\mathcal{B}}$. ☐

Remark 2. *The topology $\tau_{\mathcal{B}}$ consists of unions of basic open sets.*

Proof. The fact that basic open sets are open and $\tau_{\mathcal{B}}$ is a topology implies that the union of basic open sets are in $\tau_{\mathcal{B}}$.

On the other hand, any set U that is open in the topology generated by $\mathcal{B}$ then U is the union of the basic open sets it contains. ☐